Some characteristic properties of families of matrix monotone functions and of matrix convex functions(Banach spaces, function spaces, inequalities and their applications)

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1 Introduction

Let I be an interval in the real line R (often open, or half open) and M_n be the n by n matrix algebra. A real valued continuous function f defined in I is said to be n-(matrix) monotone if function calculus $f(a)$ and $f(b)$ for selfadjoint elements a,b of M_n with their spectrums in I preserves the order in M_n, that is,

$$a \leq b \quad \text{implies} \quad f(a) \leq f(b).$$

The function is said to be n-(matrix) convex if f keeps the convexity in M_n for any pair a and b in the same condition. Then usual classes of operator monotone functions and operator convex functions on I are expressed as the intersections of them for all n or they are the classes defined similarly on the algebra of all bounded linear operators on an infinite dimensional Hilbert space. We denote by $P_n(I)$ and $K_n(I)$ the sets of all n-monotone functions and n-convex functions (by $P_\infty(I)$ and $K_\infty(I)$ for operator monotone functions and operator convex functions, respectively). They form naturally convex cones (not linear spaces) and closed in any appropriate topologies.

These notions were introduced and discussed by K.Loewner and his two students O.Dobsch,F.Kraus more than 70 years ago but the piling structure of $P_n(I)$ and $K_n(I)$ down to $P_\infty(I)$ and $K_\infty(I)$ are investigated only recently in spite of the great necessity of these notions for many fields such as operator theory, electric networks, quantum mechanics etc... One may easily see its importance if he puts a simple question for positive matrices or operators a and b whether the relation ,$a \leq b$, implies the same relation for their square roots.

This is a half expostory article in which we show how those n-monotone functions (resp. n-convex functions) are different from usual numerical monotone (resp. convex) functions and how they look like. A main point of our discussion is to explain that certain basic properties which have been considered for a long time as characteristic ones for operator monotone (resp. convex) functions are in fact derived as the properties of just 2-monotone (resp. 2-convex) functions. Results are based on an inequality for divided differences (cf.[3]).

2 Preliminary discussions

By definition, $P_1(I)$ and $K_1(I)$ are usual numerical monotone/convex functions, and so are those functions in $P_n(I)$ and $K_n(I)$ in the numerical sense. There appear however big difference for those functions in case where $n \geq 2$. For instance, the exponential functiont e^t is a good monotone increasing function but it is not even 2-monotone, whereas its inverse function *logt* is an operator monotone function in the interval $(0, \infty)$. The functions *tlogt* and $1/t$ are known to be operator convex on the positive half line. For the basic function t^p the most well known fact is the following

Theorem (Loewner-Heinz). For $0 \leq p \leq 1$, the function t^p is operator monotone in $[0, \infty)$.

With this theorem, it has been known that if $p > 1$,or if $n \geq 2$ for integers t^p does not become even 2-monotone. For convexity, we can see that (cf [3]) t^p is 2-convex in $[0, \infty)$ if and only if $1 \leq p \leq 2$. In any case, an important point is that 2-monotonicity and 2-convexity are the turning points for this function between operator monotonicity and convexity. There is no other eventual points in the index. Moreover, we also see this kind of phenomenon in the arguments of matrix monotone/convex functions, and this is the fact that we mainly intend to enphasize in this paper.

As of now, many results are known for operator monotone/convex functions , notably their representations by integrals with respect to some unique measures. In particular, operator monotone function defined in an open interval is characterized as a Pick function. This means that it has an analytic continuation into the upper half plain which maps the half plain into itself. As consequenses, it has been known that

"Operator monotone functions on the real line R are only affine functons and operaotr convex functions on R are only quadratic".

This is the basic reason that we are used to assume the interval I being nontrivial when we discuss those functions. We shall show later that these things are already true for at the level of 2-monotone/2-convex functions far from the levels of operator monotone/convex functions.

For matirix monotone/convex functions not so many facts were known until this century, except general criteria for n-monotone functions. Even for exact gaps,

$$P_{n+1}(I) \subsetneqq P_n(I), \quad K_{n+1}(I) \subsetneqq K_n(I),$$

for every n they are believed and asserted for a long time in most of literatures with 'no' examples for $n \geq 3$(cf.[2]).

For further discussions, we need to introduce the notion of divided difference of order k, $[t_0.t_1.\ldots, t_k]$ for $k+1$-tuple of points in I. Let f be a sufficiently smooth function defined in I.

$$[t_0, t_1] = \begin{cases} \frac{f(t_1)-f(t_0)}{t_1-t_0} & \text{for } t_0 \neq t_1 \\ f'(t_0) & \text{for } t_0 = t_1 \end{cases}$$

In general,

$$[t_0, t_1, \ldots, t_k] = \begin{cases} \frac{[t_0,t_1,\ldots,t_{k-2},t_k]-[t_0,t_1,\ldots,t_{k-1}]}{t_k-t_{k-1}} & \text{for } t_{k-1} \neq t_k \\ \lim_{t'_k \to t_{k-1}} [t_0, t_1, \ldots, t_{k-1}, t'_k] & \text{for } t_{k-1} = t_k \end{cases}$$

Therefore, we have that

$$[t_0, t_0, t_0] = \frac{f''(t_0)}{2}, \qquad [t_0, t_0, t_0, t_0] = \frac{f^{(3)}(t_0)}{3!}.$$

We notice that this divided difference is permutation free, so that we can use another successive definition of divided difference.

Now we state criteria for n-monotone /n- convex functions on an open interval I, first global criterion. Let f be a function defined in I and take an n-tuple, $\{t_1, t_2, \ldots, t_n\}$ in I.

I (a) Monotonicity (Loewner 1934).

$$f \in P_n(I) \iff ([t_i, t_j]) \geq 0 \quad \text{for any } \{t_1, t_2, \ldots, t_n\}$$

. This matrix is usually called as the Loewner matrix.

I(b) Convexity (Kraus 1936)

$$f \in K_n(I) \iff ([t_1.t_i, t_j]) \geq 0 \quad \text{for any } \{t_1.t_2.\ldots, t_n\}.$$

Here t_1 can be replaced by any (fixed) t_k.

These results are established ones, but the problem is the following local criterion.

Criterion II (a). Monotonicity (Loewner 1934, Dobsch 1937-Donoghue 1974). For $f \in C^{2n-1}(I)$

$$f \in P_n(I) \iff M_n(f;t) = \left(\frac{f^{(i+j-1)}(t)}{(i+j-1)!}\right) \geq 0. \forall t \in I$$

The above matrix is a Hankel matrix. This criterion is considered as the established one but the procedure to its final conclusion has a strange story. In fact, although the proof heavily depends on the so called 'local property theorem ' stated below , whose proof is extremely hard, Loewner himself said in his paper, about this theorem, "easy and leave its proof to the readers". His student Dobsch then cited the result as "already proved one ". Forty years later Donoghue gave an almost comprehensive long proof in [1] with a little lack of rigorousity at the final stage of his proof. We have however now recognized that the proof is completed, though we are looking for a simple minded short proof.

Consider two overlapping open intervals (α, β) and (γ, δ), Suppose the function f defined in the interval (α, δ) is n-monotone on those intervals, then it is n-monotone on the interval (α, δ).

The above formulation looks quite simple. We have been however unable to prove the version of n-convex functions for $n \geq 3$. Therefore, the following (expected) criterion has not been established yet.

II (b) Convexity (Hansen-Tomiyama [2]). For $f \in C^{2n}(I)$,

$$f \in K_n(I) \Longleftrightarrow K_n(f;t) = \left(\frac{f^{(i+j)}(t)}{(i+j)!} \right) \geq 0. \forall t \in I.$$

In this formulation the necessity is fully proved in [2] and [3] but because of lack of the local property theorem we have shown only a partial sufficiency. Namely what we can assert is the result: if there exists a point t_0 such that $K_n(f;t_0)$ is positive, then there exists a neighborhood of t_0 on which f is n-convex. In order to extend this conclusion to the whole interval we have to paste these kind of results, and this is the meaning of the local property theorem.

It should be noticed here that though we have the above criteria it is not so easy to check positive semi-difiniteness of those relevant matrices in general. Actually, for 2×2 matrices this checking is rather easy and we can apply these criteria for such function t^p. But even for a 3×3 matrix its entries are all functions involving derivatives of high orders and we have to know the behavior of its eigen-values at every point of I. This might have been the reason why in a so long time examples to show the exact gaps between those classes $P_{n+1}(I)$ and $P_n(I)$ ($K_{n+1}(I)$ and $K_n(I)$ as well) are not specified for the case $n \geq 3$. As of now however we have found deep relationship between the gap problem and the (truncated) power moment problem, and by making use of this relation we can provide abundant examples (polynomials) of gaps for every n (cf.[2],[6]).

For 2 by 2 matrices, the implication from I(a) to II(a) is rather easy. By using determinants instead of matrices, just subtracting each column and row we obtain the extended Loewner determinant. We then assume that $t_1 = t_2$, which implies the non-negative property of the determinant, and it is enough to obtain the conclusion. For the case $n \geq 3$, things are not so easy and this makes the implication,$I(b) \rightarrow II(b)$ much complicated. In the above formulations the differentiability condition is not so restrictive. For, there is the way called 'regularization' which means that for any given ε we can find the C^∞ function f_ε defined on a little narrowed interval having the same property (i.e. monotonicity/convexity) and converging to f uniformly on any subinterval. This is a standard way by the molifier function used often in many fields such as in the theory of partial differential equations.

The results for the function t^p mentioned before can be easily verified by these criteria.

3 Main results

The following result is already known. Let I be an open interval.

Theorem 3.1 *([1, p.73-74]) If $f \in C^3(I)$ and $f'(t) > 0$ for every t in I, then the following assertions are equivalent.*

(1) f is 2-positive,

(2) The matrix $([t_i, t_j])$ is positive semi-definite for $\forall\{t_1, t_2\}$ in I,

(3) The matrix $M_2(f;t)$ is positive semi-definite in I,

(4) There exists a positive concave function $c(t)$ such that $f'(t) = 1/c(t)^2$ for every t in I.

Here the condition for $f'(t)$ is not so restrictive. For, if there exists a point t_0 where $f'(t_0) = 0$ it is known that f must be constant.

However, the following simple corollary of the above result had not been observed before and the result itself was derived ,in usual literature, for an operator monotone function as a consequence of its integral representation.

Corollary 3.2 *If $I = R$, then $f'(t)$ becomes constant, hence f is an affine functiion.*

Proof. Because a positive concave function defined in the whole real line R has to be constant in its geometrical figure.

We can now show the following characterization of a 2-convex function. In the theorem , although we impose the condition that $f \in C^4(I)$ the result implies with the regularization process mentined before that the local property theorem holds for an arbitrary 2-convex functions.

Theorem 3.3 *([2]) If $f \in C^4(I)$ and $f''(t) > 0$ for every t in I, then the following assertions are equivalent.*

(1) f is 2-convex,

(2) The matrix $([t_1, t_i, t_j])$ is positive semi-definite for $\forall\{t_1, t_2\}$ in I,

(3) $[t_1, t_1, t_1][t_2, t_2, t_2] \geq [t_1, t_1, t_2][t_1, t_2, t_2]$,

(4) The matrix $K_2(f; t)$ is positive semi-definite for every t in I,

(5) There exists a positive concave function $c(t)$ such that $f''(t) = 1/c(t)^3$.

Here the condition for $f''(t)$ is not too restrictive because it is known that if there exists a point t_0 such that $f''(t_0) = 0$ f must be an affine function. We leave a detailed proof of this theorem to the reference [2].

We mention a simple observation as a corollary. As in the same situation as above, the result was known before as a consequence for an operator convex function but it is in fact the result followed from 2-convexity.

Corollary 3.4 *If $I = R$, then $f''(t)$ becomes constant, hence f is a quadratic function.*

The reason is the same as the previous corollary since in this case $c(t)$ must be constant.

No such commplete characterizations have ever been known even for 3×3 matrices. We have however a general inequality for divided differences which are closely related with the above results.

Theorem 3.5 *Suppose that $f \in C^n(I)$ and $f^{(n)}(t) > 0$ for every t in I. If the function $c(t) = 1/f^{(n)}(t)^{1/n+1}$, that is, $f^{(n)}(t) = 1/c(t)^{n+1}$ is concave, then*

$$[t_1, t_2, \ldots, t_{n+1}] \leq \prod_{i=1}^{n+1} [t_i, t_i, \ldots, t_i]^{\frac{1}{n+1}},$$

where in the right member of the above inequality t_i repeats $n+1$ times.

When the function $c(t)$ is convex the inequality is reversed.

This is proved by making use of the expression of a divided difference by iterated integrals invented by Hermite long before. We leave its detailed proof to [2].

Consider the case $n = 1$,that is $f'(t) = 1/c(t)^2$. Then we have

$$0 \leq [t_1, t_2] \leq \sqrt{[t_1, t_1][t_2, t_2]} \Rightarrow [t_1, t_2]^2 \leq [t_1, t_1][t_2, t_2].$$

This implies the assertion (2) in Theorem 3.1 because the inequality shows the relevant determinant is non-negative.

In the case $n = 2$, $f"(t) = 1/c(t)^3$, from which we can see by the above inequality,

$$0 \leq [t_1, t_1, t_2] \leq [t_1, t_1, t_1]^{2/3}[t_2, t_2, t_2]^{1/3}$$

and

$$0 \leq [t_1, t_2, t_2] \leq [t_1, t_1, t_1]^{1/3}[t_2, t_2, t_2]^{2/3}.$$

Hence multiplying both sides we obtain the implication from (5) to (3) in Theorem 3.3. Thus, the inequality contributes proofs of both theorems . We may expect the inequality for $n = 3$ could bring some insight for a characterization of 3-monotone functions, but we still do not know the meaning of this inequality even for this case.

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